Review on the eigenvalue method for the system: $x' = Ax$, where A is a 2x2 matrix.

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$

Solution: $\vec{\mathbf{x}} = c_1\vec{\mathbf{x}}_1 + c_2\vec{\mathbf{x}}_2 + \cdots$,
where $\vec{\mathbf{x}}_i$ are fundamental solutions
from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

Example. Consider a 2×2 matrix $\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 5 & -3 \end{bmatrix}$. Find a general solution to the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

ANS: $0 = |A - \lambda I| = \begin{vmatrix} -1-\lambda & -2 \\ 5 & -3-\lambda \end{vmatrix} = (\lambda+1)(\lambda+3) + 10 = \lambda^2 + 4\lambda + 13$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$$

Consider $\lambda = -2+3i$ and find its eigenvector

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow (A - (-2+3i)I)\vec{v} = \begin{bmatrix} \overset{1-3i}{\cancel{-1+2-3i}} & -2 \\ 5 & \overset{-1-3i}{\cancel{-3+2-3i}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (1-3i)a - 2b = 0 & ① \\ 5a - (1+3i)b = 0 & ② \end{cases}$$

Note $① \times \frac{1+3i}{2} = ②$

Consider ①, $\Rightarrow (1-3i)a = 2b \Rightarrow \frac{a}{b} = \frac{2}{1-3i}$

Let $a = 2$, $b = 1-3i$.

Then $\vec{v} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$ is an eigenvector to $\lambda = -2+3i$.

Then a solution to $\vec{x}' = A\vec{x}$ is

$$\vec{v}e^{\lambda t} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix} e^{(-2+3i)t} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix} e^{-2t}(\cos 3t + i \sin 3t)$$

$$= e^{-2t} \begin{bmatrix} 2\cos 3t + 2i \sin 3t \\ \cos 3t + i \sin 3t - 3i \cos 3t + 3 \sin 3t \end{bmatrix}$$

$$= \underbrace{e^{-2t}\begin{bmatrix} 2\cos 3t \\ \cos 3t + 3\sin 3t \end{bmatrix}}_{\vec{x}_1(t)} + i\,\underbrace{e^{-2t}\begin{bmatrix} 2\sin 3t \\ \sin 3t - 3\cos 3t \end{bmatrix}}_{\vec{x}_2(t)}$$

$$= e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos 3t + \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin 3t\right) + i\,e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin 3t + \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t\right)$$

Thus the general solution is

$$\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t)$$

$$= C_1 e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos 3t + \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin 3t\right) + C_2 e^{-2t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin 3t + \begin{bmatrix} 0 \\ -3 \end{bmatrix}\cos 3t\right)$$

In the Final practice

Example Let $\mathbf{x}(t)$ be the solution of the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

What is $\mathbf{x}(1)$?

ANS: The char. eqn is

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)+4 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0$$

$\Rightarrow \lambda_1 = \lambda_2 = 1$

Exercise: Check if we solve $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$, we can only find one eigenvector up to a scalar.

We use the alg. on Page 1.

We solve

$$(A - \lambda I)^2 \vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}$$

So we assume $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

Then $\vec{v}_1 = (A - \lambda I)\vec{v}_2 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Thus we have:

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t, \quad \vec{x}_2(t) = (\vec{v}_1 t + \vec{v}_2) e^t = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$$

So $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$

As $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cancel{e^0}^{\,1} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cancel{e^0}^{\,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2c_1 + c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0$ and $c_2 = 1$.

Thus $\vec{x}(t) = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$

and $\vec{x}(1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^1 = \begin{bmatrix} 3e \\ e \end{bmatrix}$

Lecture 23. Nonhomogeneous Linear Systems

Given the nonhomogeneous first-order linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

where A is an $n \times n$ constant matrix and the "nonhomogeneous term" $\mathbf{f}(t)$ is a given continuous vector-valued function.

A general solution of Eq (1) has the form

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t),$$

where

- $\mathbf{x}_c = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$ is a general solution of the associated homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$,
- $\mathbf{x}_p(t)$ is a single particular solution of the original nonhomogeneous system in (1).

Undetermined Coefficients

Example 1 Apply the method of undetermined coefficients to find a particular solution of the following system.

$$\begin{cases} x' = x + 2y + 3 \\ y' = 2x + y - 2 \end{cases}$$

ANS: We assume $\vec{x}_p(t) = \begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ for some number a, b.

Then we plug them into the system.

$$\Rightarrow \begin{cases} a' = 0 = a + 2b + 3 \\ b' = 0 = 2a + b - 2 \end{cases} \Rightarrow \begin{cases} a + 2b = -3 \Rightarrow 2a + 4b = -6 \\ \underline{2a + b = 2} \end{cases}$$

$$\Rightarrow 3b = -8 \Rightarrow b = -\frac{8}{3}, \quad \text{Then } a = -3 - 2b = -3 + \frac{16}{3} = \frac{7}{3}$$

Thus we have $\vec{x}_p = \begin{bmatrix} \frac{7}{3} \\ -\frac{8}{3} \end{bmatrix}$

Recall that if we want to find $x_p(t)$ for the equation $x'' - x = e^t$, we assume $x_p = a\,te^t$ since e^t is a solution for the homogeneous equation $x'' - x = 0$.

Similarly, in general cases, we need to check the solution for $\mathbf{x}_c$ for the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

For example,

Example 2 Apply the method of undetermined coefficients to find a particular solution of the following system.

$$x' = 2x + y + 2e^t$$
$$y' = x + 2y - 3e^t$$

$\vec{x}' = A\vec{x} + \vec{f}(t)$, $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} e^t$.

Try this (exercises) Assuming ~~$\vec{x}_p(t) = \begin{bmatrix} a \\ b \end{bmatrix} e^t$~~

Why this cannot work ?

We consider the homogeneous part

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$$

$\Rightarrow \lambda = 1$ or $\lambda = 3$.

So $\vec{v}_1 e^t$ (and $\vec{v}_2 e^{3t}$) appear in the solution to the homogenous part $\vec{x}' = A\vec{x}$.

We assume $\vec{x}_p(t) = \underset{\uparrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}{\vec{a}} e^t + \underset{\uparrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}{\vec{b}} t e^t$

Then $\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t \Rightarrow \vec{x}_p'(t) = \begin{bmatrix} (a_1 + b_1)e^t + b_1 te^t \\ (a_2 + b_2)e^t + b_2 te^t \end{bmatrix}$

Plug $\vec{x}_p$ and $\vec{x}_p'$ into the system, we get

$$\begin{cases} (a_1+b_1)e^t + b_1te^t = (2a_1+a_2)e^t + (2b_1+b_2)te^t + 2e^t \\ (a_2+b_2)e^t + b_2te^t = (a_1+2a_2)e^t + (b_1+2b_2)te^t - 3e^t \end{cases}$$

Compare the coefficients for e^t, te^t, we have

$$\begin{cases} a_1+b_1-2a_1-a_2-2=0 & \Rightarrow \quad -a_1+b_1-a_2-2=0 \\ b_1-2b_1-b_2=0 & \Rightarrow \quad -b_1-b_2=0 \\ a_2+b_2-a_1-2a_2+3=0 & \Rightarrow \quad -a_2+b_2-a_1+3=0 \\ b_2-b_1-2b_2=0 & \Rightarrow \quad -b_1-b_2=0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = \frac{1}{2} \\ a_2 = 0 \\ b_1 = \frac{5}{2} \\ b_2 = -\frac{5}{2} \end{cases}$$

Then $\vec{x}_p = \vec{a}e^t + \vec{b}te^t$

$$\Rightarrow \vec{x}_p = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t + \begin{bmatrix} \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix} te^t$$

To write down the general solution to this nonhomogeneous equation, we apply the usual steps of solving the homogeneous system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

The eigenvalue and eigenvector for $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ are

$\lambda_1 = 3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Thus $\mathbf{x}_c = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Therefore, the general solution to the given system is

$$\boxed{\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{7}{3} \\ -\frac{8}{3} \end{pmatrix}.}$$

Further directions on solving nonhomogenous linear systems

Similar to Lecture 14 on solving Nonhomogeneous Equations of Second Order, there is a version of the method of **variation of parameters** in solving nonhomogenous linear systems of the following form:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

We will refer to the section 7.9 in the book by Boyce, DiPrima and Meade for this topic.